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The aerodynamic noise of small-perturbation subsonic flows

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The method of matched asymptotic expansions is used to simplify calculations of noise produced by aerodynamic flows involving small perturbations of a stream of non-negligible subsonic Mach number. This technique is restricted to problems for which the dimensionless frequency ϵ, defined as $\omega b/a_0$, is small, ω being the circular frequency, b the typical body dimension, and a_0 the speed of sound. By combining Lorentz and Galilean transformations, the problem is transformed to a space in which the approximation appropriate to the inner region is found to be incompressible flow and that appropriate to the outer, classical acoustics. This approximation for the inner region is the unsteady counterpart of the Prandtl–Glauert transformation, but is not identical to use of that transformation in a straightforward quasi-steady manner. For wings in oscillatory motion, it is the same approximation as was given by Miles (1950).

To illustrate the technique, two examples are treated, one involving a pulsating cylinder in a stream, the other the impinging of plane sound waves upon an elliptical wing in a stream.

Introduction

The method of matched asymptotic expansions has been applied to a class of problems of aerodynamic noise by Müller & Obermeier in a series of papers (Müller & Obermeier 1967, Obermeier 1967*a*, *b*). The category of problems treated by them involves, as the significant small parameter, the Mach number of the aerodynamic flow that produces the sound field, or, what is equivalent, the ratio of the characteristic wavelength of the aerodynamic motion to the acoustic wavelength of the sound produced. The power of the Müller–Obermeier technique lies in its ability to handle complicated aerodynamic flows, such as those that do not fall within any small-perturbation category. Besides this advantage, their method might be said to formalize an approximation that has seemed intuitive to earlier investigators (e.g. Lamb 1932, pp. 500, 531), i.e. to use an incompressible flow approximation to the aerodynamic flow and to join it at greater distances to a classical acoustic field. But this evaluation of the Müller–Obermeier method would overlook the important fact that it also provides a composite solution that is superior in accuracy to either the inner (incompressible) or outer (acoustic) approximation alone.

In this paper we shall exploit the matched asymptotic expansions in a somewhat different way, confining ourselves to small-perturbation flows and dispensing with the limitations to small Mach numbers. The small parameter, which determines 'inner' and 'outer' regions requiring different expansions, will now be the ratio of body dimension (or equivalent characteristic length) to the acoustic wavelength. Since, in principle, a small-perturbation problem can always be treated by acoustic methods, e.g. by distributions of acoustic singularities (Lighthill 1962), our approximation, which seems to be new, is only a labour-saving device. We find, however, that in a variety of problems, some of which we shall present here as examples, the amount of labour saved is considerable, and often results are obtained that possess all the accuracy that is desired.

Theory

We begin our treatment, therefore, with the equation that describes small perturbations of a uniform, inviscid, compressible stream whose Mach number is M and whose speed of sound is a_0; this well-known equation is

$$\frac{1}{a_0^2}\frac{\partial^2\phi}{\partial t^2}+\frac{2M}{a_0}\frac{\partial^2\phi}{\partial x\,\partial t}+M^2\frac{\partial^2\phi}{\partial x^2}=\nabla^2\phi, \tag{1}$$

where ϕ is the perturbation velocity potential and the x axis lies in the direction of the undisturbed stream. A technique for finding solutions of this equation was employed by Küssner (1940) (see also Sears 1954, 1960); it consists of making a Galilean transformation in order to reduce (1) to the wave equation and then a Lorentz transformation, which does not alter the wave equation but re-introduces the relative motion of the stream and the sound-producing elements, such as bodies.

Let the Galilean transformation be

$$x'=x-Ma_0t,\quad y'=y,\quad z'=z,\quad t'=t, \tag{2}$$

and the Lorentz transformation be

$$\left.\begin{aligned} X&=(x'+Ma_0t')/\beta,\quad Y=y',\quad Z=z',\\ T&=(t'+Mx'/a_0)/\beta,\end{aligned}\right\} \tag{3}$$

where β^2 denotes $1-M^2$.

Combining these transformations, we find

$$x=\beta X,\quad y=Y,\quad z=Z,\quad t=(T-MX/a_0)/\beta. \tag{4}$$

and, as stated,

$$\frac{1}{a_0^2}\phi_{TT}=\phi_{XX}+\phi_{YY}+\phi_{ZZ}. \tag{5}$$

The pressure perturbation $p-p_0$, which we shall call δp, is easily found to be

$$\begin{aligned}\delta p&=-\rho_0\phi_{t'}=-\rho_0\{\phi_t+Ma_0\phi_x\}\\ &=-\rho_0\{\phi_T+Ma_0\phi_X\}/\beta.\end{aligned} \tag{6}$$

Sound is sometimes produced by motions of solid bodies in the air; for such cases the kinematic boundary condition at the body surface is $DF/Dt=0$, where the body surface is described by $F(x, y, z, t) = 0$. For slender bodies this is, approximately,

$$F_t + Ma_0 F_x + \phi_y F_y + \phi_z F_z = 0. \tag{7}$$

Expressed in the transformed plane, this reads

$$\beta\mathscr{F}_T + Ma_0\{(M/\beta a_0)\mathscr{F}_T + \beta^{-1}\mathscr{F}_X\} + \phi_Y\mathscr{F}_Y + \phi_Z\mathscr{F}_Z = 0,$$

or

$$\beta^{-1}\mathscr{F}_T + \beta^{-1}Ma_0\mathscr{F}_X + \phi_Y\mathscr{F}_Y + \phi_Z\mathscr{F}_Z = 0, \tag{8}$$

where

$$\mathscr{F}(X, Y, Z, T) = F(x, y, z, t).$$

In some problems it will be desirable to interpret boundary condition (8) in physical terms in the X, Y, Z, T space. We shall, however, postpone this; for the present it must suffice to remark that the transformation we have carried out in (4)–(8) constitutes the generalization of the familiar Prandtl–Glauert transformation of steady subsonic flow to the unsteady case. It does not lead to a much simpler boundary-value problem in the transformed space, in general, and presumably that is why it has not been used. In problems of the category that we are treating here, however, further simplifications are permissible, as we shall now show.

Our matched-asymptotic-expansion technique is based upon the idea that the length that characterizes the region of flow near the body is the body dimension, say b, while the length that characterizes the distant parts of the flow is the acoustic wavelength, which is $2\pi a_0/\omega$ if ω is the (circular) frequency of the motion or the reciprocal of the characteristic time of the motion. We will consider problems in which the ratio of these two lengths is small; i.e.

$$\epsilon \equiv \omega b/a_0 \ll 1.$$

This means that gradients of physical quantities are much greater near the body, established as they are by the boundary conditions, and different approximations to the governing differential equation are permissible in the two regions. It is important to note that these features will occur in both the physical (x, y, z, t) and the transformed (X, Y, Z, T) spaces, according to (4).

The restriction to $\omega b/a_0 \ll 1$ is, of course, that of the theory of Lighthill (1962, bottom of page 155).

The process of comparing orders of magnitude, for small ϵ, of the terms of (5) is simple and is most easily accomplished by introducing dimensionless variables (Müller & Obermeier 1967), referring the spatial co-ordinates to b in an inner region and to b/ϵ in an outer region. The result, for the inner region, is

$$\phi^i_{XX} + \phi^i_{YY} + \phi^i_{ZZ} = O(\epsilon^2). \tag{9}$$

For the outer region (5) remains unchanged, since all of its terms are of the same order in ϵ.

Thus, neglecting terms of order ϵ^2 in comparison with 1, one employs an incompressible-flow approximation in the inner region of the X, Y, Z, T space, and matches it to the acoustic approximation in the outer region. Recalling the transformation used, (4), we see that the approximate differential equation for the inner region of the original x, y, z, t space is the Prandtl–Glauert equation.

Nevertheless, the approximation being suggested for this region is not just a straightforward quasi-steady Prandtl–Glauert approximation. The difference is that T and not t appears in our boundary condition, (8). On the body surface, T and t differ by quantities of order ϵ, which would be omitted in a 'straightforward quasi-steady approximation'.

We see that, for problems involving small ϵ, there is a useful extension of the Prandtl–Glauert method to unsteady flows. We therefore proceed to interpret the transformed boundary-value problem, as promised above. To do so, one must multiply (8) by β, to give

$$\mathscr{F}_T + Ma_0 \mathscr{F}_X + \beta\phi_Y \mathscr{F}_Y + \beta\phi_Z \mathscr{F}_Z = 0. \tag{10}$$

This is the boundary condition for flow at speed Ma_0 past the Prandtl–Glauert-stretched body $\mathscr{F} = 0$, provided that $\beta\phi$ is the perturbation velocity potential. The body is performing a distorted version of the motion of the original body in time, because T is $T(x, t)$ according to (4). The boundary-value problem for the region near the body is therefore the problem of incompressible flow at speed Ma_0 past this unsteady thin body.

All this is not quite new. Miles (1950) studied the matter of the 'compressibility correction' for wings oscillating sinusoidally in subsonic flow. Although his derivation is rather different from ours, relates specifically to sinusoidal oscillations of planar bodies, and is expressed in different terminology, it seems clear that his 'correction rule' is the same as what we have derived here.

With this confirmation of our conclusions for the inner region, we are ready to proceed with the matching of inner and outer approximations, which we shall carry out in two illustrative examples. There is, however, one more detail to be accomplished in order to facilitate this matching, namely to introduce appropriate dimensionless co-ordinates for the respective regions. Let

$$\left.\begin{aligned}
&\varphi^i = \phi^i/\omega b^2, \quad \varphi^o = \phi^o/\omega b^2,\\
&\xi^i = X/b, \quad \eta^i = Y/b, \quad \zeta^i = Z/b,\\
&\xi^o = \epsilon\xi^i, \quad \eta^o = \epsilon\eta^i, \quad \zeta^o = \epsilon\zeta^i,\\
&r^i = (\xi^{i2} + \eta^{i2} + \zeta^{i2})^{\frac{1}{2}}, \quad r^o = \epsilon r^i,\\
&\tau = \omega T.
\end{aligned}\right\} \tag{11}$$

The differential equation for φ^i is then Laplace's equation in the ξ^i, η^i, ζ^i space, neglecting $O(\epsilon^2)$, while the differential equation for φ^o, from (5), becomes

$$\varphi^o_{\tau\tau} = \nabla^2\varphi^o, \tag{12}$$

where the right-hand side is the Laplacian in the ξ^o, η^o, ζ^o space.

Examples

Noise produced by a pulsating cylinder in a stream

As our first example, let us treat the two-dimensional case of a symmetrical thin cylinder pulsating symmetrically in a stream. Let its ordinates be given by

$$y = \pm bg(x)\, e^{i\omega t} \quad \text{for} \quad -b \leqslant x \leqslant b, \tag{13}$$

so that boundary condition (7) is simply

$$v(x,y,t) = \{i\omega g(x) + Ma_0 g'(x)\} b\, e^{i\omega t} \tag{14}$$

(for the upper surface), where v denotes ϕ_y and, for the thin cylinder, $v(x,y,t)$ is to be replaced by $v(x,+0,t)$. Thus, the boundary condition in the X, Y, T space, (8), is

$$V(X,+0,T) = \{i\omega g(\beta X) + Ma_0 g'(\beta X)\} b \exp\{i\omega(T - MX/a_0)/\beta\} \tag{15}$$

for $-b/\beta \leqslant X \leqslant b/\beta$, where V denotes ϕ_Y.

The perturbation potential for the inner region can be written down immediately, for the solution of Laplace's equation that satisfies this boundary condition at the slit $-b/\beta \leqslant X \leqslant b/\beta$, $Y = \pm 0$, is obtained by a distribution of sources of strength $2V(X, +0, T)$, as is well known; thus

$$\varphi^i = \frac{1}{\pi\omega b}\int_{-1/\beta}^{1/\beta} V(bu, +0, \tau/\omega) \ln[b^2(\xi^i - u)^2 + b^2\eta^{i2}]\, du. \tag{16}$$

This is the inner approximation. To match it to the outer, it is necessary to expand it in a form appropriate for large r^i and express the result in terms of r^o; this yields the approximate expression for the outer solution for small r^o (Van Dyke 1964). In the interests of brevity, we shall not reproduce the details of this expansion. (It is facilitated by integrating by parts the term containing g'.) The result is, neglecting $O(\epsilon^2)$,

$$(\varphi^o)^i = (\varphi^i)^o = \frac{1}{\pi\beta^2} e^{i\tau/\beta}\Big\{A[(i/\beta)\ln r^o + M\xi^o/r^{o2}] + \epsilon B\left[(M/\beta^2)\ln r^o - (1-M^2)\frac{i\xi^o}{\beta^2 r^{o2}} - \frac{M}{\beta}\left(\frac{1}{r^{o2}} - 2\frac{\xi^{o2}}{r^{o4}}\right)\right]\Big\}, \tag{17}$$

where A denotes $\int_{-b}^{b} g(x)\,dx/b$

and B denotes $\int_{-b}^{b} g(x)\,x\,dx/b.$

According to (17), the inner flow field, at its outer edges, is equivalent to an oscillating source, dipole, and quadrupole. In this embodiment, Van Dyke's matching procedure simply calls for the replacement of these three incompressible singularities by the corresponding acoustic ones:

$$\varphi^o = \Big\{C_1[J_0(r^o/\beta) - iY_0(r^o/\beta)] + C_2\frac{\xi^o}{r^o}[J_1(r^o/\beta) - iY_1(r^o/\beta)] + \frac{\epsilon C_3}{r^o}[J_1(r^o/\beta) - iY_1(r^o/\beta)] - \frac{\epsilon C_3 \xi^{o2}}{\beta r^{o2}}[J_2(r^o/\beta) - iY_2(r^o/\beta)]\Big\} e^{i\tau/\beta}, \tag{18}$$

where the coefficients are determined from (17) by use of the expansions of the Bessel functions for small argument; they are found to be

$$C_1 = -\tfrac{1}{2}A/\beta^3 + \tfrac{1}{2}i\epsilon BM/\beta^5,$$

$$C_2 = -\tfrac{1}{2}iAM/\beta^3 - \tfrac{1}{2}\epsilon B(1+M^2)/\beta^5, \quad C_3 = \tfrac{1}{2}iBM/\beta^4. \tag{19}$$

The solution is now complete; φ^i and φ^o can be put back into dimensional form, brought back into the original x, y, t space, and the composite solution, valid for all r, can be formed according to the formula

$$\phi = \phi^i + \phi^o - (\phi^i)^o. \tag{20}$$

Reflexion of sound waves from a wing in a stream

As our second example let us consider plane sound waves impinging upon a wing. Let their direction of propagation be normal to the plane of the wing as sketched in figure 1. We think of this as an idealization of practical problems relating to the effectiveness of a wing as a baffle against sound propagation. While many problems have been solved involving such sound baffles in air at rest, we believe that the corresponding phenomena in the presence of a stream at non-negligible Mach number are not known.

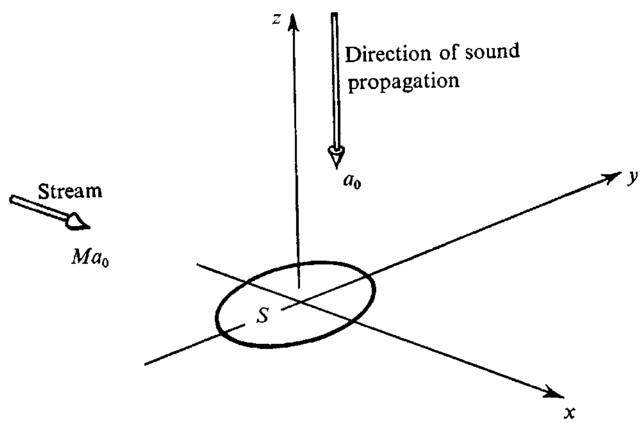

FIGURE 1

We shall consider here an elliptical wing whose major and minor axes are of length b and βb, respectively. This particular choice of dimensions has the obvious advantage that the wing becomes circular in the X, Y, Z, T space.

Suppose the propagating sound waves have frequency $\omega/2\pi$; the incident perturbation field is then described by

$$\phi_1 = A \exp [i\omega(t + z/a_0)], \tag{21}$$

and we shall assume again that $\epsilon = \omega b/a_0 \ll 1$, so that our method of matched asymptotic expansions is applicable.

Let ϕ denote the additional velocity potential due to the presence of the wing; the boundary condition for ϕ is then the condition that $\partial\phi/\partial z$ cancel the impinging perturbation velocities at $z = 0$ on S; i.e.

$$\phi_z(x, y, 0, t) = -(Ai\omega/a_0) \exp (i\omega t) \quad \text{on } S \tag{22}$$

and
$$\phi_Z(X, Y, 0, T) = -(Ai\omega/a_0) \exp [i\omega(T - MX/a_0)/\beta] \tag{23}$$

within the circle of radius b and centre at the origin in the X, Y plane.

Although, in principle, we could carry the terms of order ϵ in this boundary condition, as in the preceding example, we propose to neglect them here for simplicity. This means that we omit the term MX/a_0 in (23), whereupon the boundary condition becomes $\phi_Z = -(Ai\omega/a_0)\exp(i\omega T)$ in the circle. The solution of this potential problem is available (Lamb 1932, p. 144): viz.

$$\left.\begin{aligned} \varphi^i &= -2(iA/a_0 b)\,e^{i\tau/\beta}[\tan^{-1}(1/\theta) - 1/\theta]\,\zeta^i, \\ \text{where}\qquad \theta^2 &= -\tfrac{1}{2} + \tfrac{1}{2}r^{i2}\{1 + (1 - 2/r^{i2} + 1/r^{i4} + 4\zeta^{i2}/r^{i4})^{\frac{1}{2}}\} \end{aligned}\right\} \tag{24}$$

and other notation is the same as in (11).

The expansion of this solution for large r^i is

$$(\varphi^i)^o = (2iA/3\pi a_0 b)\,(\zeta^i/r^{i3})\,e^{i\tau/\beta}. \tag{25}$$

Hence, the matching procedure calls only for an acoustic dipole for the outer flow field in this case. It is found to be

$$\varphi^o = -(2A/3\pi a_0 b)\,(\epsilon^2\zeta^o/r^{o2})\,[1/\beta - i/r^o]\,e^{i(\tau - r^o)/\beta} \tag{26}$$

or, when expressed in dimensional form and returned to the x, y, z, t space,

$$\phi^o = -\frac{2\beta^2}{3\pi}\frac{A}{a_0}\,\omega\,\frac{b^3 z}{\sigma^2}\left[\frac{\omega}{\beta a_0} - \frac{i\beta}{\sigma}\right]\exp\left[i\omega\left(t + \frac{Mx}{\beta^2 a_0} - \frac{\sigma}{\beta^2 a_0}\right)\right], \tag{27}$$

where σ denotes $\{x^2 + \beta^2(y^2 + z^2)\}^{\frac{1}{2}}$.

Again the calculation is to be completed by forming the composite solution, $\phi^i + \phi^o - (\phi^i)^o$, and subsequently the pressure by means of the first of (6). The latter is a tedious calculation, and we shall not carry it out here, but turn instead to a more interesting subject.

The effect of circulation about the wing

The solution obtained above is the one that involves no circulation about the wing. This was correct in Lamb's problem, the solution to which was exploited above, for reasons of symmetry. But the problem being attacked here is not symmetrical; the elliptical wing possesses leading and trailing edges, and one must ask whether fluctuating circulation would not be produced by the velocity fluctuations ϕ_z of (22).

For sufficiently high frequencies (large enough ϵ), this question might be moot, but, since we are considering only small ϵ and subsonic flow, we believe that the answer is clear: circulation would be produced, and the Kutta–Joukowsky condition provides an approximation to determine its magnitude. Shen & Crimi (1965), among others, have considered the question of the limits of validity of the Kutta–Joukowsky condition, by treating an oscillating airfoil in a viscous fluid. Their conclusion is that as long as the 'reduced frequency', which is ϵ/M in our notation, is less than or equal to 1 the classical Kutta–Joukowsky condition should be applied at every instant. Their criterion is satisfied in the case we are treating.

The solution obtained above is therefore deficient and must be augmented by consideration of the effects of fluctuating circulation about the wing. This affords an opportunity to demonstrate further the utility of our method of approximation, for it serves to relate this problem to one of an oscillating wing with circulation in an incompressible flow. For example, the fluctuating lift of a circular wing carrying out oscillations perpendicular to its plane has been calculated, in the unsteady-thin-airfoil approximation, by Schade & Krienes (1947). We shall now show how this result (the lift) alone can be used to approximate to the sound field. Actually, the sound field that results from this fluctuating lift is an order of magnitude larger than what we have calculated above.

It may be of value to review our theory at this point and to relate what will follow to the general philosophy adopted previously. So far, our method has been to concentrate on the perturbation velocity potential ϕ and to construct its composite approximation by matching its inner and outer forms. This procedure has been adopted because of the fact that aerodynamic flows are usually described in terms of ϕ or its derivatives and their boundary conditions are usually given in these terms. It is also possible to work directly with the pressure perturbation δp; in the small-perturbation cases we are treating, it satisfies the same differential equations as does ϕ. If the fluctuating force is given, as in the problem we are now attacking, it will be the more convenient quantity to use, as Lighthill (1962) has done. What we want to make clear is that to obtain inner and outer approximations to δp and to construct from these a composite solution is equivalent to doing the same for ϕ.

Now, the problem of the reflexion of plane sound waves by an elliptical wing has been reduced, above, to a harmonic boundary-value problem in X, Y, Z, T, with the boundary condition given in (23), and subsequent matching to an acoustic field. The Schade–Krienes calculation represents a solution of this harmonic boundary-value problem, but it does not give us ϕ in detail, only the resulting lift. It is interesting to notice that, whereas ϕ is independent of the stream speed (as are the results given in (24)–(26)) the lift surely depends on this parameter; so does the pressure perturbation. According to (6), we want to consider the oscillating wing in a stream of speed Ma_0, and then carry the pressure perturbation back to x, y, z, t in the manner prescribed by (6), i.e. with a factor $1/\beta$. Thus, the Schade–Krienes problem that is pertinent to our problem is that of a circular wing of radius b in an incompressible flow of speed Ma_0. In (23) we see that it oscillates with frequency ω/β. Schade & Krienes' result can be expressed as follows:

$$L(T) = -2{\cdot}813\rho_0 Ma_0 b^2 \phi_Z(T), \tag{28}$$

where $L(T)$ is the lift and $\phi_Z(T)$ denotes the value of ϕ_Z on the wing, which is given by (23). Again, we shall neglect $O(\epsilon)$, which means that we omit the term MX/a_0 in (23). Thus, $L(T)$ has the form $K \exp(i\omega t/\beta)$.

The flow field in X, Y, Z, T, at large r^i, is therefore the field due to an oscillating concentrated force of this magnitude. The corresponding pressure-perturbation field is known (Lighthill 1962) to be that of a pressure-dipole. Let the pressure perturbation in X, Y, Z, T be denoted by δP, so that $\delta p = \delta P/\beta$; then

$$(\delta P^i)^o = (K\zeta^i/4\pi b^2 r^{i3})\, e^{i\tau/\beta}. \tag{29}$$

The outer approximation is therefore a corresponding acoustic pressure-dipole:

$$\delta P^o = (iK/4\pi b^2)\,(\epsilon^2\zeta^o/r^{o2})\,[1/\beta - i/r^o]\, e^{i(\tau - r^o)/\beta}. \tag{30}$$

Let us carry this pressure-perturbation formula back to the x, y, z, t space, express it in dimensional co-ordinates, and write out the constant K, which is $2{\cdot}813\rho_0 Mb^2 Ai\omega$. The result is

$$\delta p^o = -(2{\cdot}813/4\pi) A\rho_0 M\beta b^2\omega \frac{z}{\sigma^2}\left[\frac{\omega}{\beta a_0} - \frac{i\beta}{\sigma}\right]\exp\left[i\omega\left(t + \frac{Mx}{\beta^2 a_0} - \frac{\sigma}{\beta^2 a_0}\right)\right], \tag{31}$$

which is consistent with a result of Lighthill (1962, p. 165).

At large radii, this contribution to the pressure field is $O(\epsilon^2)$, whereas the pressure perturbation derived from (27), without wing circulation, is found to be $O(\epsilon^3)$. Unfortunately, our calculation has not provided us with an inner approximation corresponding to (31). The efficacy of the wing as a noise baffle has to be deduced from (31) for small σ (where, of course, the approximation of (29) holds). Curves of constant pressure-amplitude in the combined field of the impinging waves and the wing effect are, roughly, dipole-like loops above and below the wing. The effect seems rather small, since the amplitude is affected only about 1% at distances of about $\sqrt{(10)}\,b$ above and below the wing.

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